Formule di Teoria dei Segnali

L.Verdoliva

Formule di trigonometria

$$
\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha \\
\cos^2\alpha &= \frac{1+\cos 2\alpha}{2} \\
\sin^2\alpha &= \frac{1-\cos 2\alpha}{2} \\
\sin 2\alpha &= 2\sin\alpha\cos\alpha \\
\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha \\
\cos\alpha\cos\beta &= \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \\
\sin\alpha\sin\beta &= \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)] \\
\sin\alpha\cos\beta &= \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]
\end{aligned}
$$

Formule di Eulero

$$
\cos\alpha = \frac{e^{j\alpha}+e^{-j\alpha}}{2} \qquad \sin\alpha = \frac{e^{j\alpha}-e^{-j\alpha}}{2j} \qquad e^{j\alpha} = \cos\alpha + j\sin\alpha
$$

Media temporale per segnali aperiodici (1) e per segnali periodici (2)

$$
\begin{aligned}
(1) \quad < x(t) > &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt & < x(n) > &= \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N} x(n) \\
(2) \quad < x(t) > &= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt & < x(n) > &= \frac{1}{N_0}\sum_{n=0}^{N_0-1} x(n)
\end{aligned}
$$

a) *Invarianza temporale* $y(t) = x(t-t_0) \implies < y(t) >=< x(t) >$

$y(n) = x(n-n_0) \implies < y(n) >=< x(n) >$

b) *Linearità* $z(\cdot) = a\,x(\cdot) + b\,y(\cdot) \implies < z(\cdot) >= a < x(\cdot) > + b < y(\cdot) >$

Potenza per segnali aperiodici (1) e per segnali periodici (2) ed Energia (3)

$$
\begin{aligned}
(1) \quad P_x &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt & P_x &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \\
(2) \quad P_x &= \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} |x(t)|^2 \, dt & P_x &= \frac{1}{N_0} \sum_{n=0}^{N_0-1} |x(n)|^2 \\
(3) \quad E_x &= \int_{-\infty}^{+\infty} |x(t)|^2 \, dt & E_x &= \sum_{n=-\infty}^{+\infty} |x(n)|^2
\end{aligned}
$$

Potenza ed Energia per segnali noti

a) *Segnale costante*	$x(t) = A$	$\Longrightarrow$	$P_x = A^2$
b) *Gradino unitario*	$x(t) = A\,\mathrm{u}(t)$	$\Longrightarrow$	$P_x = A^2/2$
c) *Signum*	$x(t) = A\,\mathrm{sign}(t)$	$\Longrightarrow$	$P_x = A^2$
d) *Segnale sinusoidale*	$x(t) = A\,\cos(2\pi f_0 t + \theta)$	$\Longrightarrow$	$P_x = A^2/2$
e) *Impulso rettangolare*	$x(t) = A\,\Pi(t/T)$	$\Longrightarrow$	$E_x = A^2 T$
f) *Impulso triangolare*	$x(t) = A\,\Lambda(t/T)$	$\Longrightarrow$	$E_x = \frac{2}{3} A^2 T$
g) *Esponenziale monolatero*	$x(t) = A\,e^{-t/T}\mathrm{u}(t)$	$\Longrightarrow$	$E_x = \frac{A^2 T}{2}$
h) *Impulso rettangolare discreto*	$x(n) = A\,\mathcal{R}_N(n)$	$\Longrightarrow$	$E_x = A^2\,N$
i) *Esponenziale monolatero discreto*	$x(n) = A\,a^n\mathrm{u}(n)$ con $\lvert a\rvert < 1$	$\Longrightarrow$	$E_x = A^2 \frac{1}{1-a^2}$
j) *Gradino unitario discreto*	$x(n) = x(n) = A\,\mathrm{u}(n)$	$\Longrightarrow$	$P_x = A^2/2$
k) *Segnale alternato*	$x(n) = A\,(-1)^n$	$\Longrightarrow$	$P_x = A^2$
l) *Segnale sinusoidale discreto*	$x(n) = A\,\cos(2\pi\nu_0 n)$	$\Longrightarrow$	$P_x = A^2/2$

Potenza ed Energia mutua

$$
\begin{aligned}
P_{xy} &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t)\, dt & P_{xy} &= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y^*(n) \\
E_{xy} &= \int_{-\infty}^{+\infty} x(t)\, y^*(t)\, dt & E_{xy} &= \sum_{n=-\infty}^{+\infty} x(n)\, y^*(n)
\end{aligned}
$$

a) *Invarianza temporale* $y(t) = x(t - t_0) \implies P_y = P_x$ e $E_y = E_x$

$y(n) = x(n - n_0) \implies P_y = P_x$ e $E_y = E_x$

b) *Non Linearità* $z(\cdot) = x(\cdot) + y(\cdot) \implies P_z = P_x + P_y + 2\,\mathrm{Re}[P_{xy}]$

$\implies E_z = E_x + E_y + 2\,\mathrm{Re}[E_{xy}]$

<u>Funzione di autocorrelazione</u> per segnali di potenza aperiodici (1) e periodici (2) e per segnali di energia (3)

$$(1) \quad R_x(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, x^*(t-\tau)\, dt \qquad R_x(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x^*(n-m)$$

$$(2) \quad R_x(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, x^*(t-\tau)\, dt \qquad R_x(m) = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n)\, x^*(n-m)$$

$$(3) \quad R_x(\tau) = \int_{-\infty}^{+\infty} x(t)\, x^*(t-\tau)\, dt \qquad R_x(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x^*(n-m)$$

<u>Funzione di mutua correlazione</u> per segnali di potenza (1) e per segnali di energia (2)

$$(1) \quad R_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, y^*(t-\tau)\, dt \qquad R_{xy}(m) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, y^*(n-m)$$

$$(2) \quad R_{xy}(\tau) = \int_{-\infty}^{+\infty} x(t)\, y^*(t-\tau)\, dt \qquad R_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y^*(n-m)$$

a) *Valore nell'origine* $R_x(0) = \begin{cases} E_x \\ P_x \end{cases}$ $R_{xy}(0) = \begin{cases} E_{xy} \\ P_{xy} \end{cases}$

b) *Simmetria coniugata* $R_x(\cdot) = R_x^*(-(\cdot))$ $R_{xy}(\cdot) = R_{yx}^*(-(\cdot))$

c) *Limitatezza* $|R_x(\cdot)| \le R_x(0)$ $|R_{xy}(\cdot)| \le \begin{cases} \sqrt{E_x E_y} \\ \sqrt{P_x P_y} \end{cases}$

<u>Formule di utilità</u>

$$\sum_{n=0}^{+\infty} \alpha^n = \frac{1}{1-\alpha} \quad |\alpha| < 1 \qquad \sum_{n=M}^{N} \alpha^n = \begin{cases} \frac{\alpha^M - \alpha^{N+1}}{1-\alpha} & \alpha \neq 1 \\ N - M + 1 & \alpha = 1 \end{cases}$$

Proprietà $\delta(t)$ e $\delta(n)$

$\int_{t_1}^{t_2} x(t)\,\delta(t)\,dt = \begin{cases} x(0) & 0 \in (t_1, t_2) \\ 0 & \text{altrimenti} \end{cases}$	$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{altrimenti} \end{cases}$
$\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$	$\sum_{k=-\infty}^{+\infty} \delta(n-k) = 1$
$\int_{-\infty}^{+\infty} x(t)\delta(t-t_0)\,dt = x(t_0)$	$\sum_{n=-\infty}^{+\infty} x(n)\delta(n-n_0) = x(n_0)$
$x(t)\,\delta(t) = x(0)\,\delta(t)$	$x(n)\,\delta(n) = x(0)\,\delta(n)$
$x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)$	$x(n)\,\delta(n-n_0) = x(n_0)\,\delta(n-n_0)$
$x(t) * \delta(t-t_0) = x(t-t_0)$	$x(n) * \delta(n-n_0) = x(n-n_0)$
$x(t-t_1) * \delta(t-t_2) = x[t-(t_1+t_2)]$	$x(n-n_1) * \delta(n-n_2) = x[n-(n_1+n_2)]$
$\delta(t) = \delta(-t)$	$\delta(n) = \delta(-n)$
$\int_{-\infty}^{+\infty} x(\alpha)\delta(t-\alpha)\,d\alpha = x(t) * \delta(t) = x(t)$	$\sum_{k=-\infty}^{+\infty} x(k)\delta(n-k) = x(n) * \delta(n) = x(n)$
$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \mathrm{u}(t) \quad \leftrightarrow \quad \delta(t) = \frac{d\mathrm{u}(t)}{dt}$	$\sum_{k=-\infty}^{n} \delta(k) = \mathrm{u}(n) \quad \leftrightarrow \quad \delta(n) = \mathrm{u}(n) - \mathrm{u}(n-1)$
$\delta(at) = \frac{1}{\lvert a\rvert}\delta(t)$	

Sistemi LTI nel dominio del tempo

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} x(\alpha)\,h(t-\alpha)\,d\alpha \\ &= x(t) * h(t) \end{aligned} \qquad\qquad y(n) = \sum_{k=-\infty}^{+\infty} x(k)\,h(n-k) = x(n) * h(n)$$

a) *Proprietà commutativa* $x(\cdot) * h(\cdot) = h(\cdot) * x(\cdot)$

b) *Proprietà distributiva* $x(\cdot) * [h_1(\cdot) + h_2(\cdot)] = x(\cdot) * h_1(\cdot) + x(\cdot) * h_2(\cdot)$

c) *Proprietà associativa* $x(\cdot) * [h_1(\cdot) * h_2(\cdot)] = [x(\cdot) * h_1(\cdot)] * h_2(\cdot)$

d) *Proprietà associativa mista* $a[x(\cdot) * h(\cdot)] = [ax(\cdot)] * h(\cdot) = x(\cdot) * [ah(\cdot)]$

e) *Invarianza temporale* $x(t-t_1) * h(t-t_2) = y(t-(t_1+t_2))$

$x(n-n_1) * h(n-n_2) = y(n-(n_1+n_2))$

Sistema non dispersivo	$\Longleftrightarrow$	$h(\cdot) = k\delta(\cdot)$	
Sistema causale	$\Longleftrightarrow$	$h(t) = 0$ per $t < 0$	$h(n) = 0$ per $n < 0$
Sistema stabile	$\Longleftrightarrow$	$\int_{-\infty}^{+\infty} \lvert h(t)\rvert \, dt < \infty$	$\sum_{n=-\infty}^{+\infty} \lvert h(n)\rvert < \infty$

Serie di Fourier

$$\textit{Sintesi} \quad x(t) = \sum_{k=-\infty}^{+\infty} X_k \, e^{j2\pi k f_0 t} \qquad x(n) = \sum_{k=0}^{N_0-1} X_k \, e^{j2\pi k \nu_0 n}$$

$$\textit{Analisi} \quad X_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \, e^{-j2\pi k f_0 t} dt \qquad X_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x(n) \, e^{-j2\pi k \nu_0 n}$$

$$x(\cdot) \text{ reale} \longrightarrow X_{-k} = X_k^* \longleftrightarrow \begin{cases} |X_{-k}| = |X_k| \\ \angle X_{-k} = -\angle X_k \end{cases}$$

1) *Linearità* $z(\cdot) = ax(\cdot) + by(\cdot) \longleftrightarrow Z_k = aX_k + bY_k$

2) *Traslazione temporale* $y(t) = x(t - t_0) \longleftrightarrow Y_k = X_k \, e^{-j2\pi k f_0 t_0}$

$y(n) = x(n - n_0) \longleftrightarrow Y_k = X_k \, e^{-j2\pi k \nu_0 n_0}$

3) *Riflessione* $y(\cdot) = x(-\cdot) \longleftrightarrow Y_k = X_{-k}$

4) *Derivazione* $y(t) = \frac{dx(t)}{dt} \longleftrightarrow Y_k = j2\pi k f_0 X_k$

5) *Differenza prima* $y(n) = x(n) - x(n-1) \longleftrightarrow Y_k = (1 - e^{-j2\pi k \nu_0}) X_k$

6) *Relazione di Parseval*

$\frac{1}{T_0} \int_{T_0} |x(t)|^2 \, dt = \sum_{k=-\infty}^{+\infty} |X_k|^2 \qquad \frac{1}{N_0} \sum_{<N_0>} |x(n)|^2 = \sum_{k=<N_0>} |X_k|^2$

Trasformata di Fourier

$$\textit{Sintesi} \quad x(t) = \int_{-\infty}^{+\infty} X(f) \, e^{j2\pi f t} \, df \qquad x(n) = \int_{-1/2}^{1/2} X(\nu) \, e^{j2\pi \nu n} \, d\nu$$

$$\textit{Analisi} \quad X(f) = \int_{-\infty}^{+\infty} x(t) \, e^{-j2\pi f t} \, dt \qquad X(\nu) = \sum_{n=-\infty}^{+\infty} x(n) \, e^{-j2\pi \nu n}$$

$$x(\cdot) \text{ reale} \quad \longrightarrow \quad X(-(\cdot)) = X^*(\cdot) \quad \longleftrightarrow \quad \begin{cases} |X(-(\cdot))| = |X(\cdot)| \\ \angle X(-(\cdot)) = -\angle X(\cdot) \end{cases}$$

1) *Linearità*	$a_1 x_1(\cdot) + a_2 x_2(\cdot)$	$\longleftrightarrow$	$a_1 X_1(\cdot) + a_2 X_2(\cdot)$
2) *Riflessione*	$x(-(\cdot))$	$\longleftrightarrow$	$X(-(\cdot))$
3) *Dualità*	$x(t)$	$\longleftrightarrow$	$X(f)$
	$X(t)$	$\longleftrightarrow$	$x(-f)$
4) *Cambiamento di scala*	$x(at)$	$\longleftrightarrow$	$\frac{1}{\lvert a\rvert} X\left(\frac{f}{a}\right)$
5) *Espansione*	$x\left[\frac{n}{M}\right]$	$\longleftrightarrow$	$X(M\nu)$
6) *Decimazione*	$x(Mn)$	$\longleftrightarrow$	$\frac{1}{M}\sum_{k=0}^{M-1} X\left(\frac{\nu-k}{M}\right)$
7) *Convoluzione*	$x(\cdot) * y(\cdot)$	$\longleftrightarrow$	$X(\cdot)Y(\cdot)$
8) *Prodotto*	$x(t)y(t)$	$\longleftrightarrow$	$X(f) * Y(f)$
	$x(n)y(n)$	$\longleftrightarrow$	$X(\nu) * Y(\nu) = \int_{-\frac{1}{2}}^{\frac{1}{2}} X(u)Y(\nu-u)\,du$
9) *Derivazione d.d.t*	$\frac{d^k x(t)}{dt^k}$	$\longleftrightarrow$	$(j2\pi f)^k X(f)$
10) *Differenza prima*	$x(n) - x(n-1)$	$\longleftrightarrow$	$(1 - e^{-j2\pi\nu})X(\nu)$
11) *Derivazione d.d.f*	$t^k x(t)$	$\longleftrightarrow$	$\left(\frac{j}{2\pi}\right)^k \frac{d^k X(f)}{df^k}$
12) *Integrazione*	$\int_{-\infty}^{t} x(\alpha)\,d\alpha$	$\longleftrightarrow$	$\frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
13) *Somma corrente*	$\sum_{k=-\infty}^{n} x(k)$	$\longleftrightarrow$	$\frac{X(\nu)}{1-e^{-j2\pi\nu}} + \frac{1}{2}X(0)\widetilde{\delta}(\nu)$
14) *Traslazione d.d.t*	$x(t-t_0)$	$\longleftrightarrow$	$X(f)\,e^{-j2\pi f t_0}$
	$x(n-n_0)$	$\longleftrightarrow$	$X(\nu)\,e^{-j2\pi\nu n_0}$
15) *Traslazione d.d.f*	$x(t)\,e^{j2\pi f_0 t}$	$\longleftrightarrow$	$X(f-f_0)$
	$x(n)\,e^{j2\pi\nu_0 n}$	$\longleftrightarrow$	$X(\nu-\nu_0)$
16) *Modulazione*	$x(t)\cos(2\pi f_0 t + \theta)$	$\longleftrightarrow$	$\frac{1}{2}X(f-f_0)e^{j\theta} + \frac{1}{2}X(f+f_0)e^{-j\theta}$
	$x(t)\cos(2\pi\nu_0 n + \theta)$	$\longleftrightarrow$	$\frac{1}{2}X(\nu-\nu_0)e^{j\theta} + \frac{1}{2}X(\nu+\nu_0)e^{-j\theta}$

17) *Campionamento d.d.f*	$\sum_{n=-\infty}^{+\infty} x(t-nT)$	$\longleftrightarrow$	$\sum_{k=-\infty}^{+\infty} \frac{1}{T} X\left(\frac{k}{T}\right) \delta\left(f-\frac{k}{T}\right)$
	$\sum_{k=-\infty}^{+\infty} x(n-kn)$	$\longleftrightarrow$	$\sum_{k=0}^{N-1} \frac{1}{N} X\left(\frac{k}{N}\right) \widetilde{\delta}\left(f-\frac{k}{N}\right)$
18) *Campionamento d.d.t*	$\sum_{n=-\infty}^{+\infty} x(nT)\delta(t-nT)$	$\longleftrightarrow$	$\sum_{k=-\infty}^{+\infty} \frac{1}{T} X\left(f-\frac{k}{T}\right)$
	$\sum_{k=-\infty}^{+\infty} x(kN)\delta(n-kN)$	$\longleftrightarrow$	$\sum_{k=0}^{N-1} \frac{1}{N} X\left(f-\frac{k}{N}\right)$
19) *Valore nell'origine*	$X(0) = \int_{-\infty}^{+\infty} x(t)\,dt$		$x(0) = \int_{-\infty}^{+\infty} X(f)\,df$
	$X(0) = \sum_{n=-\infty}^{+\infty} x(n)$		$x(0) = \int_{-1/2}^{+1/2} X(\nu)\,d\nu$
20) Coeff. Fourier per $x_p(t)$	$x_p(t) = \sum_{n=-\infty}^{+\infty} x(t-nT)$		$X_k = A_k\, e^{j\theta_k} = \frac{1}{T} X\left(\frac{k}{T}\right)$

21) *Relazione di Parseval*

$$\int_{-\infty}^{+\infty} |x(t)|^2\,dt = \int_{-\infty}^{+\infty} |X(f)|^2\,df \qquad \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \int_{-1/2}^{+1/2} |X(\nu)|^2\,d\nu$$

<u>Trasformate notevoli</u> (segnali tempo continuo)

1) *Impulso rettangolare*	$A\,\mathrm{rect}\left(\frac{t}{T}\right)$	$\longleftrightarrow$	$AT\,\mathrm{sinc}(fT)$
2) *Impulso triangolare*	$A\,\Lambda\left(\frac{t}{T}\right)$	$\longleftrightarrow$	$AT\,\mathrm{sinc}^2(fT)$
3) *Esponenziale monolatero*	$A\,e^{-t/T}\mathrm{u}(t)$	$\longleftrightarrow$	$\frac{AT}{1+j2\pi fT}$
	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}\mathrm{u}(t)$	$\longleftrightarrow$	$\frac{1}{(\alpha+j2\pi f)^n}$
4) *Esponenziale bilatero*	$A\,e^{-\lvert t\rvert/T}$	$\longleftrightarrow$	$\frac{2AT}{1+(2\pi fT)^2}$
5) *Funzione sinc*	$A\,\mathrm{sinc}(2Bt)$	$\longleftrightarrow$	$\frac{A}{2B}\,\mathrm{rect}\left(\frac{f}{2B}\right)$
6) *Funzione sinc^2*	$A\,\mathrm{sinc}^2(2Bt)$	$\longleftrightarrow$	$\frac{A}{2B}\,\Lambda\left(\frac{f}{2B}\right)$
7) *Impulso ideale*	$\delta(t)$	$\longleftrightarrow$	1
8) *Impulso ideale traslato*	$\delta(t-t_0)$	$\longleftrightarrow$	$e^{-j2\pi ft_0}$
9) *Segnale costante*	A	$\longleftrightarrow$	$A\,\delta(f)$
10) *Gradino unitario*	$\mathrm{u}(t)$	$\longleftrightarrow$	$\frac{1}{j2\pi f} + \frac{1}{2}\delta(f)$

11) *Funzione signum*	$\text{sign}(t)$	$\longleftrightarrow$	$\frac{1}{j\pi f}$
12) *Fasore*	$A\,e^{j2\pi f_0 t}$	$\longleftrightarrow$	$A\,\delta(f-f_0)$
13) *Segnale coseno*	$A\cos(2\pi f_0 t)$	$\longleftrightarrow$	$\frac{A}{2}\,\delta(f-f_0)+\frac{A}{2}\,\delta(f+f_0)$
14) *Segnale seno*	$A\sin(2\pi f_0 t)$	$\longleftrightarrow$	$\frac{A}{2j}\,\delta(f-f_0)-\frac{A}{2j}\,\delta(f+f_0)$
15) *Treno di impulsi*	$\sum_{n=-\infty}^{+\infty}\delta(t-nT)$	$\longleftrightarrow$	$\frac{1}{T}\sum_{k=-\infty}^{+\infty}\delta\left(f-\frac{k}{T}\right)$

<u>Trasformate notevoli</u> (segnali tempo discreto)

1) *Impulso rettangolare*	$\mathcal{R}_N(n)$	$\longleftrightarrow$	$\begin{cases} N & \nu=0,\pm1,\pm2,\cdots \\ \frac{\sin(\pi\nu N)}{\sin(\pi\nu)}\,e^{-j(N-1)\pi\nu} & \nu\neq0,\pm1,\pm2,\cdots \end{cases}$
2) *Impulso triangolare*	$\mathcal{B}_{2N}(n)$	$\longleftrightarrow$	$\begin{cases} N & \nu=0,\pm1,\pm2,\cdots \\ \frac{\sin^2(\pi\nu N)}{N\sin^2(\pi\nu)}\,e^{-j(N+1)\pi\nu} & \nu\neq0,\pm1,\pm2,\cdots \end{cases}$
3) *Esponenziale monolatero*	$a^n\text{u}(n)$	$\longleftrightarrow$	$\frac{1}{1-ae^{-j2\pi\nu}}$
4) *Esponenziale bilatero*	$a^{\lvert n\rvert}$	$\longleftrightarrow$	$\frac{1-a^2}{1-2a\cos(2\pi\nu)+a^2}$
5) *Funzione sinc*	$2\nu_c\,\text{sinc}(2\nu_c n)$	$\longleftrightarrow$	$\text{rep}_1\left[\text{rect}\left(\frac{\nu}{2\nu_c}\right)\right]$
6) *Funzione sinc*^{2}	$2\nu_c\,\text{sinc}^2(2\nu_c n)$	$\longleftrightarrow$	$\text{rep}_1\left[\Lambda\left(\frac{\nu}{2\nu_c}\right)\right]$
7) *Impulso ideale*	$\delta(n)$	$\longleftrightarrow$	1
8) *Impulso ideale traslato*	$\delta(n-n_0)$	$\longleftrightarrow$	$e^{-j2\pi\nu n_0}$
9) *Segnale costante*	A	$\longleftrightarrow$	$A\,\widetilde{\delta}(\nu)$
10) *Gradino unitario*	$\text{u}(n)$	$\longleftrightarrow$	$\frac{1}{1-e^{-j2\pi\nu}}+\frac{1}{2}\widetilde{\delta}(\nu)$
11) *Funzione signum*	$\text{sign}(n)$	$\longleftrightarrow$	$\frac{2}{1-e^{-j2\pi\nu}}$
12) *Fasore*	$A\,e^{j2\pi\nu_0 n}$	$\longleftrightarrow$	$A\,\widetilde{\delta}(\nu-\nu_0)$
13) *Segnale coseno*	$A\cos(2\pi\nu_0 n)$	$\longleftrightarrow$	$\frac{A}{2}\,\widetilde{\delta}(\nu-\nu_0)+\frac{A}{2}\,\widetilde{\delta}(\nu+\nu_0)$
14) *Segnale seno*	$A\sin(2\pi\nu_0 n)$	$\longleftrightarrow$	$\frac{A}{2j}\,\widetilde{\delta}(\nu-\nu_0)-\frac{A}{2j}\,\widetilde{\delta}(\nu+\nu_0)$
15) *Treno di impulsi*	$\sum_{k=-\infty}^{+\infty}\delta(n-kN)$	$\longleftrightarrow$	$\frac{1}{N}\sum_{k=-\infty}^{+\infty}\delta\left(\nu-\frac{k}{N}\right)$

Filtri RC e CR

1) *filtro RC*	$h(t) = \frac{1}{RC}\, e^{-\frac{t}{RC}}\, \mathrm{u}(t)$	$\longleftrightarrow$	$H(f) = \frac{1}{1+j2\pi fRC}$
2) *filtro CR*	$h(t) = \delta(t) - \frac{1}{RC}\, e^{-\frac{t}{RC}}\, \mathrm{u}(t)$	$\longleftrightarrow$	$H(f) = \frac{j2\pi fRC}{1+j2\pi fRC}$
3) *Rapp. in decibel*	$\lvert H(f)\rvert_{dB} = 10\log_{10}\frac{\lvert H(f)\rvert^2}{\lvert H(f_0)\rvert^2}$		
4) *filtro RC (dB)*	$\lvert H(f)\rvert_{dB} = -10\log_{10}[1+(2\pi fRC)^2]$	*frequenza di taglio*	$f_T = \frac{1}{2\pi RC}$

Filtraggio di segnali periodici

1)	$x(t) = \cos(2\pi f_0 t)$	$y(t) = \lvert H(f_0)\rvert\, \cos(2\pi f_0 t + \angle H(f_0))$
2)	$x(t) = A_0 + 2\sum_{n=1}^{+\infty} A_k \cos(2\pi k f_0 t + \theta_k)$	$y(t) = A_0 H(0) + 2\sum_{n=1}^{+\infty} A_k \lvert H(kf_0)\rvert \cos(2\pi k f_0 t + \theta_k + \angle H(kf_0))$

Densità spettrale per segnali di potenza aperiodici (1) e periodici (2) e segnali di energia (3)

(1) $S_x(f) = \lim_{T\to\infty}\frac{1}{T}\lvert X_T(f)\rvert^2$ (2) $S_x(f) = \sum_{k=-\infty}^{+\infty}\lvert X_k\rvert^2\delta\left(f - \frac{k}{T_0}\right)$ (3) $S_x(f) = \lvert X(f)\rvert^2$

La trasformata di Fourier della funzione di autocorrelazione è la densità spettrale

$\mathcal{R}_x(\tau) \quad \longleftrightarrow \quad S_x(f)$

Formule di Teoria dei Segnali

L.Verdoliva

Probabilità

$$
\begin{aligned}
P(A) &= \frac{\text{casi favorevoli}}{\text{casi possibili}} \quad \text{classico} \\
P(A) &= \lim_{n\to\infty} \frac{n_A}{n} \qquad\qquad \text{frequentista}
\end{aligned}
$$

Approccio assiomatico

$$
\begin{aligned}
A_1 &: \quad P(A) \geq 0, \quad \forall A \in S \\
A_2 &: \quad P(\Omega) = 1 \\
A_3 &: \quad P(A \cup B) = P(A) + P(B) \qquad \text{con } A \cap B = \emptyset \\
A_{3bis} &: \quad P(\cup_i A_i) = \sum_i P(A_i) \qquad \text{con } A_i \cap A_j = \emptyset, \forall i \neq j
\end{aligned}
$$

Proprietà e prob. condizionale

$$
\begin{aligned}
P(\emptyset) &= 0 \\
P(\overline{A}) &= 1 - P(A) \\
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A|B) &= \frac{P(AB)}{P(B)} = \frac{P(B|A)P(A)}{P(B)} \rightarrow P(AB) = P(A/B)P(B) \\
P(AB) &= P(A)P(B) \ \text{ A e B indipendenti} \\
P(A_1 \ldots A_n) &= P(A_1)P(A_2|A_1)P(A_3|A_2A_1)\ldots P(A_n|A_{n-1}\ldots A_1) \quad \text{Regola della catena} \\
P(A) &= \sum_{i=1}^{N} P(A|B_i)P(B_i), \text{ con } \Omega = \cup_{i=1}^{N} B_i, \ B_i \cap B_j = \emptyset, \ \forall i \neq j \quad \text{Prob. totale}
\end{aligned}
$$

Prove di Bernoulli

$\Omega = \{\omega_0, \omega_1\}$, con $P(\omega_0) = p, P(\omega_1) = 1 - p = q$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$P(k \text{ volte } \omega_0 \text{ su } n \text{ prove }) = \binom{n}{k} p^k \, q^{n-k}$$

$$P(\text{almeno } k \text{ volte } \omega_0 \text{ su } n \text{ prove }) = \sum_{i=k}^{n} \binom{n}{k} p^k \, q^{n-k}$$

Variabile aleatoria

$$\begin{aligned} \text{CDF} &: F_X(x) = P(X \leq x) \\ \text{pdf} &: f_X(x) = \frac{dF_X(x)}{dx} \end{aligned}$$

Proprietà CDF e pdf

$$\begin{aligned} \text{CDF} &: F(-\infty) = 0, F(+\infty) = 1 \\ & \quad x_1 < x_2 \ \rightarrow \ F(x_1) \leq F(x_2) \\ & \quad F(x^+) = F(x) \ \text{ con } \ F(x^+) = \lim_{\epsilon \to 0} F(x+\epsilon) \\ & \quad P(X > x) = 1 - P(X \leq x) = 1 - F_X(x) \\ & \quad P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) \\ & \quad P(X = x) = F_X(x) - F_X(x^-) \ \text{ con } \ F(x^-) = \lim_{\epsilon \to 0} F(x-\epsilon) \\ \text{pdf} &: f_X(x) \geq 0 \\ & \quad \int_{-\infty}^{+\infty} f_X(\alpha) d\alpha = 1 \\ & \quad \int_{x_1}^{x_2} f_X(\alpha) d\alpha = F_X(x_2) - F_X(x_1) \end{aligned}$$

Esempi di variabili aleatorie

$$\begin{aligned} X \sim \mathcal{B}(p) &: P(X=1) = p, \ P(X=0) = 1-p \\ X \sim Binom(n,p) &: P(X=k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad k = 0, 1, ..., n \\ X \sim Geo(p) &: P(X=k) = p(1-p)^{k-1} \qquad k = 1, 2, 3, ... \\ X \sim \mathcal{U}(a,b) &: f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & \text{altrimenti} \end{cases} \\ & \quad F_X(x) = \begin{cases} 0 & x \in]-\infty, a[\\ \frac{x-a}{b-a} & x \in [a,b] \\ 1 & x \in]b, +\infty[\end{cases} \\ X \sim \mathcal{N}(\mu, \sigma) &: f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ & \quad F_X(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\alpha-\mu)^2}{2\sigma^2}} d\alpha = G\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

$$
\begin{aligned}
X \sim \text{Exp}(\lambda) \quad &: \quad f_X(x) = \lambda\, e^{-\lambda x} \text{u}(x) \\
&\qquad F_X(x) = (1 - e^{-\lambda x}) \text{u}(x) \\
X \sim \text{Lap}(\lambda) \quad &: \quad f_X(x) = \frac{\lambda}{2}\, e^{-\lambda |x|} \\
&\qquad F_X(x) = \begin{cases} \frac{1}{2}\, e^{\lambda x} & x < 0 \\ 1 - \frac{1}{2}\, e^{-\lambda x} & x \geq 0 \end{cases} \\
X \sim \text{Rayleigh}(b) \quad &: \quad f_X(x) = \frac{2x}{b}\, e^{-\frac{x^2}{b}}\, \text{u}(x) \\
&\qquad F_X(x) = (1 - e^{-\frac{x^2}{b}})\, \text{u}(x)
\end{aligned}
$$

Caratterizzazione sintetica

$$
\begin{aligned}
\text{v.a. discreta} \quad &: \quad \mu_X = E[X] = \sum_{i=1}^{M} x_i p_i \\
&\qquad E[X^2] = \sum_{i=1}^{M} x_i^2 p_i \\
&\qquad \sigma_X^2 = E[(X - \mu_X)^2] = \sum_{i=1}^{M} (x_i - \mu_X)^2 p_i = E[X^2] - \mu_X^2 \\
\text{v.a. continua} \quad &: \quad \mu_X = E[X] = \int_{-\infty}^{+\infty} x\, f_X(x)\, dx \\
&\qquad E[X^2] = \int_{-\infty}^{+\infty} x^2\, f_X(x)\, dx \\
&\qquad \sigma_X^2 = E[(X - \mu_X)^2] = \int_{-\infty}^{+\infty} (x - \mu_X)^2\, f_X(x)\, dx
\end{aligned}
$$

Trasformazione lineare di v.a.

$$
\begin{aligned}
Y &= aX + b \\
\mu_Y &= E[Y] = aE[X] + b \\
\sigma_X^2 &= \text{var}[Y] = E[(Y - \mu_Y)^2] = a^2 \sigma_X^2
\end{aligned}
$$

$$
\begin{aligned}
F_Y(y) &= \begin{cases} F_X\left(\frac{y-b}{a}\right) & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right) & a < 0 \end{cases} \\
f_Y(y) &= \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)
\end{aligned}
$$

Relazioni per distribuzioni condizionali

$$
\begin{aligned}
F_X(x|B) &= \frac{P(X \le x, B)}{P(B)} \\
f_X(x|B) &= \frac{d}{dx} F_X(x|B) \\
E[x|B] &= \int_{-\infty}^{+\infty} x\, f_X(x|B)\, dx \\
F_X(x) &= \sum_{i=1}^{n} F_X(x|A_i) P(A_i),\ \text{con } \Omega = \cup_{i=1}^{N} A_i,\ A_i \cap A_j = \emptyset,\ \forall i \neq j \\
f_X(x) &= \sum_{i=1}^{n} f_X(x|A_i) P(A_i),\ \text{con } \Omega = \cup_{i=1}^{N} A_i,\ A_i \cap A_j = \emptyset,\ \forall i \neq j \\
E[X] &= \sum_{i=1}^{n} E[x|A_i] P(A_i),\ \text{con } \Omega = \cup_{i=1}^{N} A_i,\ A_i \cap A_j = \emptyset,\ \forall i \neq j
\end{aligned}
$$

Coppie di variabili aleatorie

vv.aa. discrete :
$$P(X = x_i, Y = y_j) = P(X = x_i|Y = y_j)P(Y = y_j) = P(Y = y_j|X = x_i)P(X = x_i)$$
$$P(X = x_i) = \sum_{j=1}^{M} P(X = x_i, Y = y_j)$$

X e Y indipendenti :
$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j)$$

vv.aa. continue :
$$F_{XY}(x, y) = P(X \le x, Y \le y) = P((X, Y) \in D)$$
$$f_{XY}(x, y) = \frac{\partial F(x, y)}{\partial x \partial y}, \qquad F_{XY}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(\alpha, \beta)\, d\alpha\, d\beta = \int\int_{D} f(\alpha, \beta)\, d\alpha\, d\beta$$
$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x, y)\, dx\, dy = 1$$

$$F_X(x) = F_{XY}(x, +\infty), \quad F_Y(y) = F_{XY}(+\infty, y)$$
$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dy, \quad f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dx$$

X e Y indipendenti : $f_{XY}(x, y) = f_X(x) f_Y(y)$

Caratterizzazione statistica delle coppie di variabili aleatorie

Correlazione : $R_{XY} = E[XY] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} xy\, f_{XY}(x, y)\, dx\, dy$

Covarianza : $C_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (x - \mu_X)(y - \mu_Y)\, f_{XY}(x, y)\, dx\, dy$

$$C_{XY} = R_{XY} - \mu_X \mu_Y$$

X e Y incorrelate : $C_{XY} = 0, \quad R_{XY} = \mu_X \mu_Y$

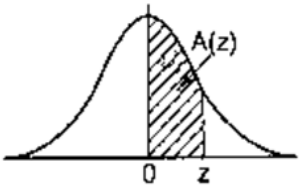
X e Y ortogonali : $R_{XY} = 0$

Coefficiente di correlazione : $\rho = \dfrac{E[(X - \mu_X)(Y - \mu_Y)]}{\sigma_X \sigma_Y} = \dfrac{C_{XY}}{\sigma_X \sigma_Y} \qquad |\rho| \leq 1$

Somma di variabili aleatorie indipendenti $Z = X + Y$

$$f_Z(z) = f_X(x) * f_Y(y)$$
$$\mu_Z = \mu_X + \mu_Y$$
$$\sigma_Z^2 = \sigma_X^2 + \sigma_Y^2$$

AREA SOTTESA DALLA FUNZIONE DI DISTRIBUZIONE NORMALE TRA Z=0 E Z



Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1571
0.4	.1554	.1591	.1628	.16Q4	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549
0.7	.2580	.2611	.2642	.2673	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4429	.4441
1.6	.4452	.4463	.4474	.4484	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4761	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4864	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916
2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4979	.4980	.4981
2.9	.4981	.4982	.4982	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.4987	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.4990	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993
3.2	.4993	.4993	.4994	.4994	.4994	.4994	.4994	.4995	.4995	.4995
3.3	.4995	.4995	.4995	.4996	.4996	.4996	.4996	.4996	.4996	.4997
3.4	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4997	.4998
3.5	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998	.4998
3.6	.4998	.4998	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.7	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.8	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999	.4999
3.9	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000
4.0	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000	.5000